HOW TO SEE A MEROMORPHIC ONE-FORM

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Abstract. Visualizing a real-valued function f:R–>R is easy–draw its graph. Visualizing a holomorphic function f:C–>C seems harder–its graph is a surface embedded in a 4-manifold. The traditional work-around is to draw a picture on the domain (like grid lines, or a bunch of rabbits), and push it forward to the range. Since the most interesting functions on the complex plane are many-to-one, one must restrict the image to a tiny region of the domain or confront a multiply covered mess (like a kandinsky, or a pile of copulating rabbits). By instead drawing the pullback of grid lines on the range, we can recover a clear picture of any meromorphic function: its branch points and poles, with all their multiplicities, are visible all at once. The same process works to draw a meromorphic one-form on any Riemann surface, and, according to Klein, were the way that Riemann himself understood them. The lines in this pullback picture can be understood as either electric field lines or trajectories in fluid flow, depending on your favorite physical interpretation, and we will follow Riemann in using the corresponding physical intuition to 'prove' the Riemann mapping theorem and Riemann-Roch.

Drawing a function $f : \mathbb{C} \to \mathbb{C}$ is hard. Our grade-school techniques for visualizing functions–draw the graph–doesn't work, since the graph sits in a four-dimensional space. My talk in a sentence: "To visualize a meromorphic function $f = u + iv$, draw the contour plot for its real part u."

1. Graphing

But first, let's talk about the traditional ways to visualize functions, say $f : \mathbb{R} \to \mathbb{R}$. First thing you do is graph your function, eg x^2, or x^3, or $x^3 - 3x$.

But as grownup mathematicians, we think of f as a map from one real line (the x-axis) to another (the y-axis). The graph is a great intermediate step for visulizing this: push the x-axis up to the graph, then the graph onto the y axis. You can imagine this as a movie, and it's great: folding and then squishing. You can also try to identify the domain and range as one copy of the real line, and see the movie as a deformation of $\mathbb{R}$ within $\mathbb{R}$: every point going to its image. Sort of okay in your head, but really impossible to show anyone else: a line sliding around within a line looks stationary.

2. Pushing forward

The movie strategy works better in 2-D. The function $z \mapsto z^2$ on the complex plane looks like the colorful figure below. There are many movies (homotopies) connecting the identity map to z^2; we could also take a straight line homotopy $z \mapsto z^2$ and choose a different part of the domain to color. Note the double cover. For more complicated functions $z^3 - 3z$, we start getting overlap. You could of course just pick one region and follow it (like with the bishops hat), but that basically requires knowing what the function looks like before you've drawn it!

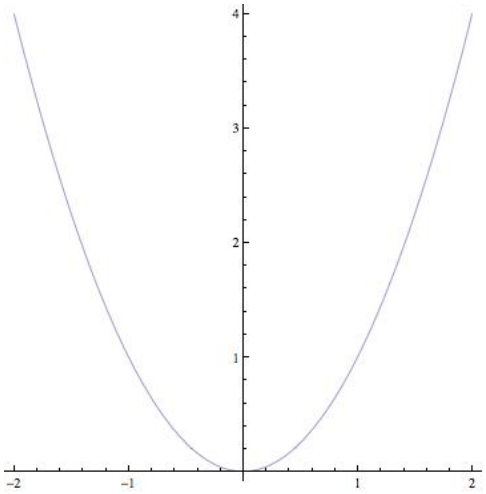

Figure 1.1. x^2

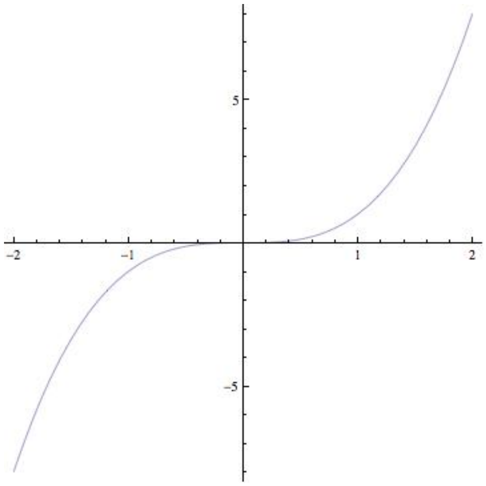

Figure 1.2. x^3

This strategy of drawing a grid (or some more fanciful picture, eg rabbits) on the domain and drawing its imagine on the range is the "pushforward" strategy. And we know from, say, differential geometry, you can't push forward a function (ie, coloring instructions) along by another function; all you can do is pull back.

3. Pulling back

Our new strategy: draw a picture on the range, and pull it back to the domain. Any outlandish picture works, but its easiest for my computer to draw (and most natural as well) evenly spaced grid lines on the range.

Consider the function $z^2 = x^2 - y^2 + i2xy$. The preimage of the blue horizontal lines in the range are the curves of constant imaginary part $2xy$ in the domain; the

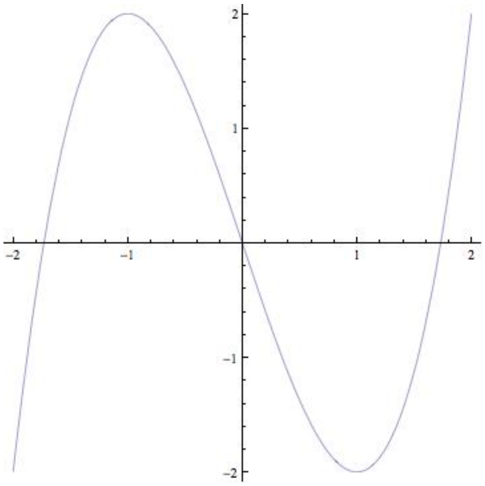

FIGURE 1.3. $x^3 - 3x$

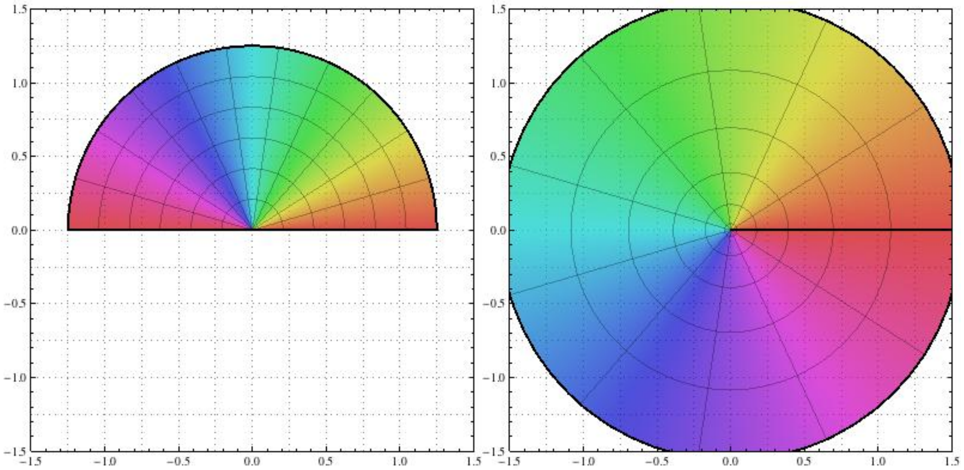

FIGURE 2.1. z^2 domain and range

preimages of the red vertical lines in the range are the curves of constant real part $x^2 - y^2$ in the domain.

From this picture, we can see the following:

- Conformality: Preimages of grid lines intersect at right angles except where $df = 0$ (cuz f is conformal)
- Ramification points: Zeroes of the derivative show up bright as day, and with their degree too.

The lines of constant imaginary part foliate $\mathbb{C}$, and the lines of constant real part form an orthogonal foliation (at least away from ramification points).

The real and imaginary parts of any holomorphic function $f = u + iv$ are harmonic, and determine one another (up to a constant) via the Cauchy-Riemann

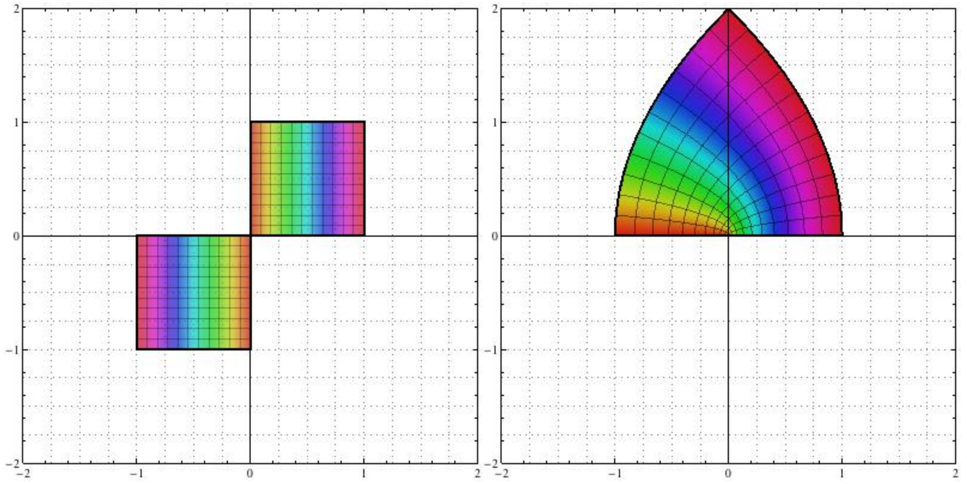

FIGURE 2.2. z^2 domain and range

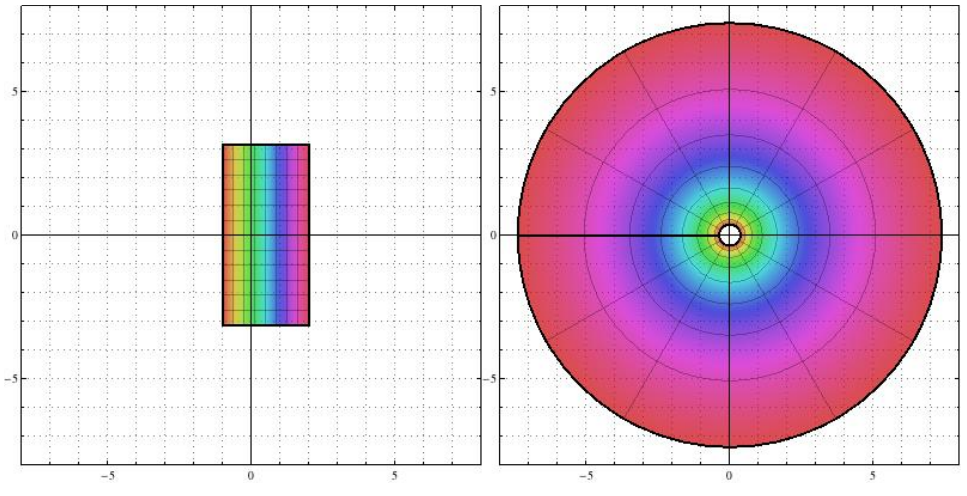

FIGURE 2.3. $\exp z$ domain and range

equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

More succinctly, $dv = Jdu = *du$ where J is the complex structure on the cotangent bundle, and $*$ is the Hodge star.

The contour plot pictures only depend on du and dv: the curves of constant real (imaginary) part are the integral curves for the line field $\mathrm{Ker} du$ ($\mathrm{Ker} dv$). Thus the picture for u entirely determines the picture for v, and we need only draw one of them.

For example, if we color the range in horizontal strips (ie, by imaginary part), we get another picture of z^2.

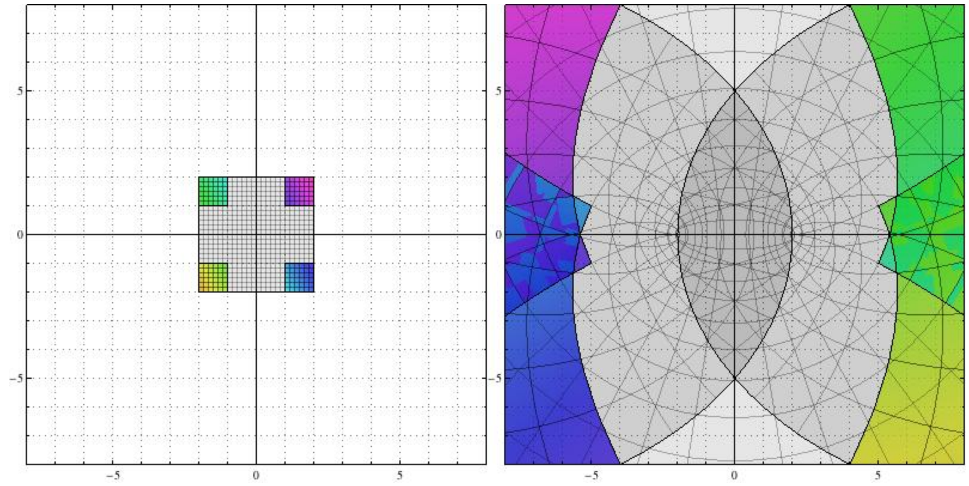

FIGURE 2.4. $z^3 - 3z$ domain and range

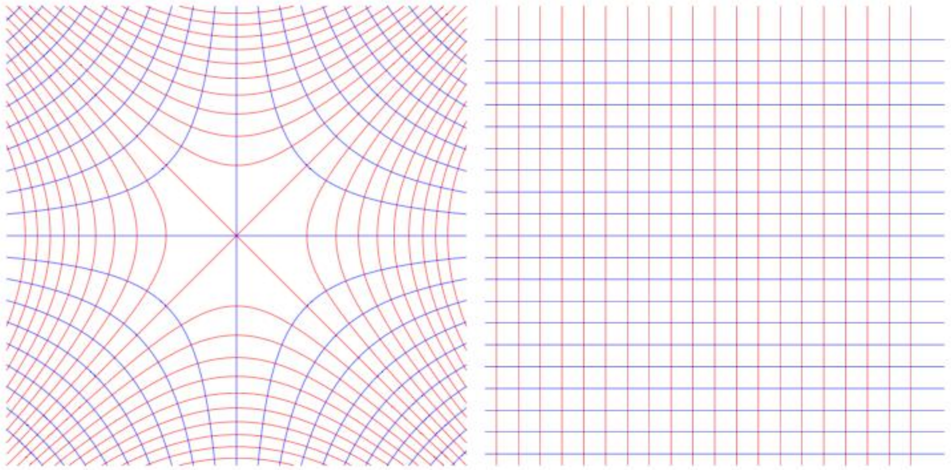

FIGURE 3.1. z^2 pullback picture, domain and range

No guessing is required here: we can use the same coloring of the range for every choice of function on the domain. So here are a bunch of examples, all using the above coloring of the range.

Note how $z^3 - 3z$ looks like z^3 at infinity, but like z^2 near each of its degree two ramification points.

4. WEIERSTRAUSS LOCAL FORM

Our pictures don't lie: away from ramification points, a holomorphic function 'looks like' a stretched copy of the standard plane, and near zeroes of the derivative, it 'looks like' our picture for $z \mapsto z^n$. This is just the Weierstrauss local form.

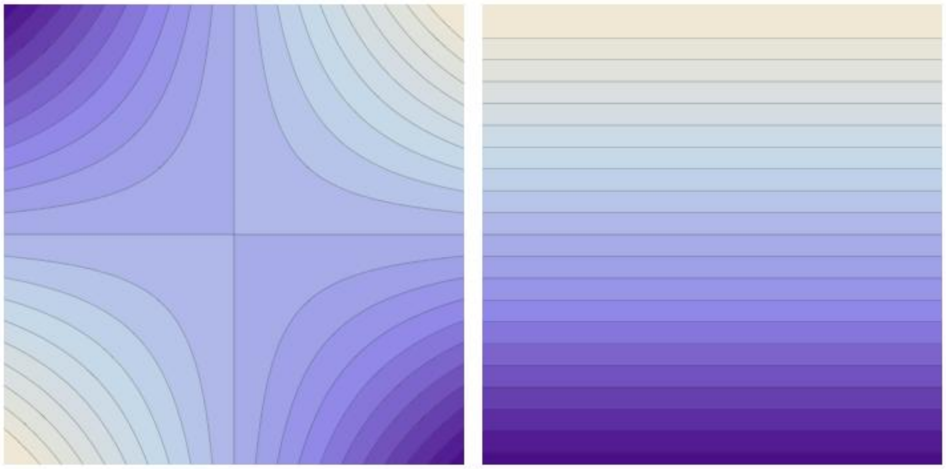

FIGURE 3.2. z^2 pullback picture, domain and range

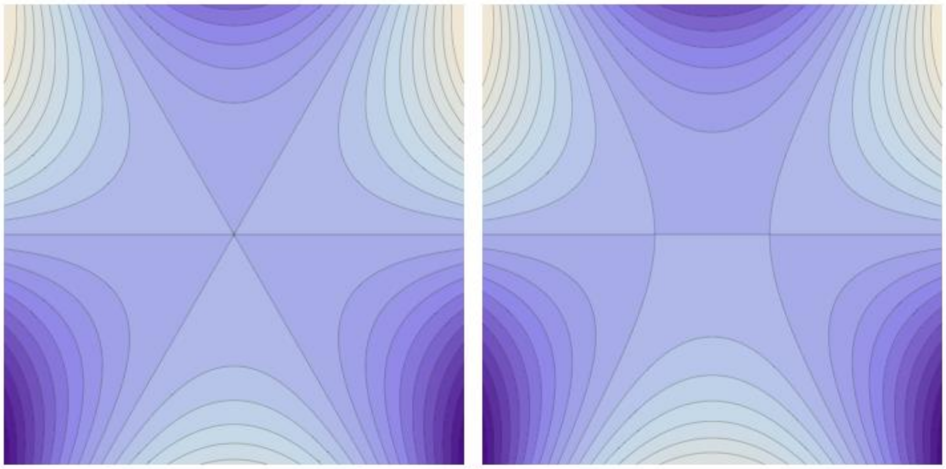

FIGURE 3.3. $z^3$and $z^3 - 3z$

Expand your analytic function

$$\begin{aligned} f(z) &= a_n z^n + a_{n+1} z^{n+1} + \cdots \\ &= z^n (a_n + a_{n+1} z + \cdots) \\ &= z^n h(z)^n \end{aligned}$$

where $h(z) = \sqrt[n]{a_n + a_{n+1} z + \cdots}$, defined in some small neighborhood of 0. Then let $w(z) = zh(z)$. Note $w(0) = 0$ and $w'(0) = \sqrt[n]{a_n} \neq 0$ (so w is conformal and biholomorphic) and $f(z) = w(z)^n$. In the neighborhood of 0, f is just $z \mapsto z^n$ after a conformal change of coordinates. This works just as well for n negative, which we'll see when discussing poles.

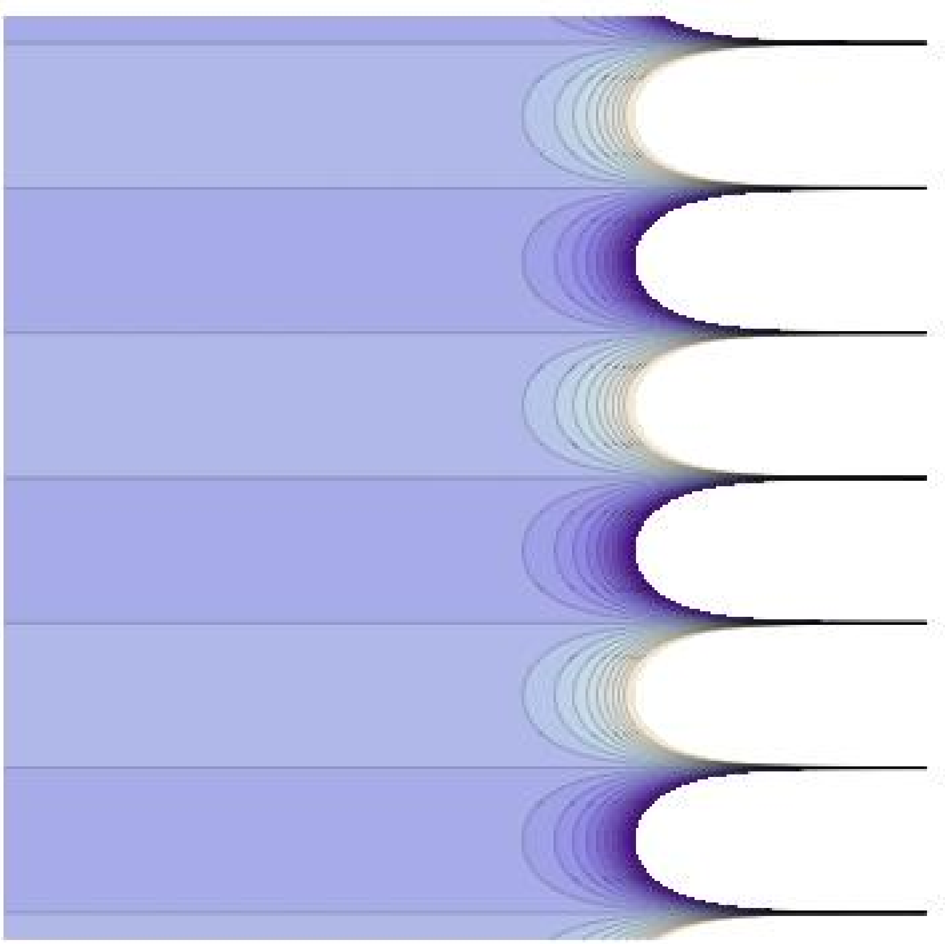

FIGURE 3.4. $\exp z$

5. INFINITIES

What does $1/z$ look like? Well, it should look like the picture for the identity map z *in the neighborhood of infinity* on the Riemann sphere. Since each horizontal line on the plane is a circle passing through the point at infinity, we see a bunch of nested circles (which should remind you of the electric field for a dipole). The white petals are the values of imaginary part too large or small to be colored by one of the twenty horizontal bands on the range.

Of course, $1/z^2$ is just the square of $1/z$, so its picture is the pullback of that for $1/z$ along the squaring map, which just doubles everything by folding up at the origin.

Howbout an essential singularity? We know it has to attain all values infinitely often in the neighborhood of the pole. But nothing crazy is going on, see the beautiful picture of $e^{1/z}$. (Which looks like the picture for e^z viewed in a nbhd of infinity, of course).

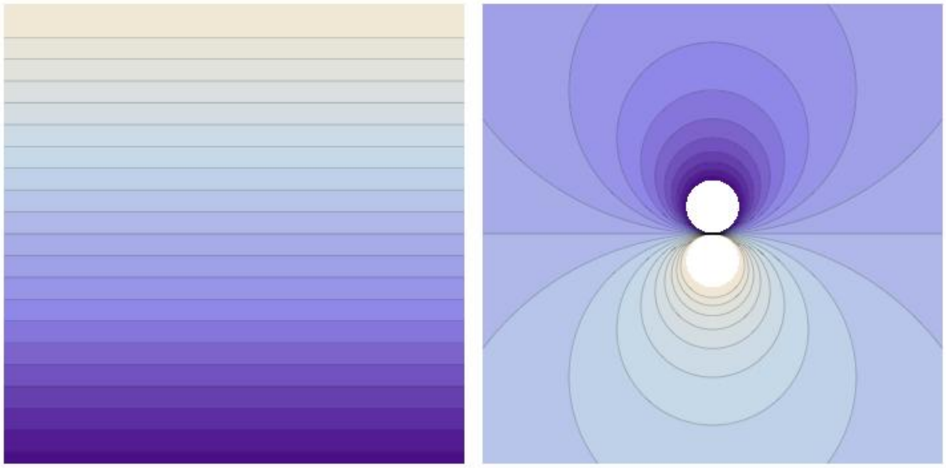

FIGURE 5.1. z and $1/z$

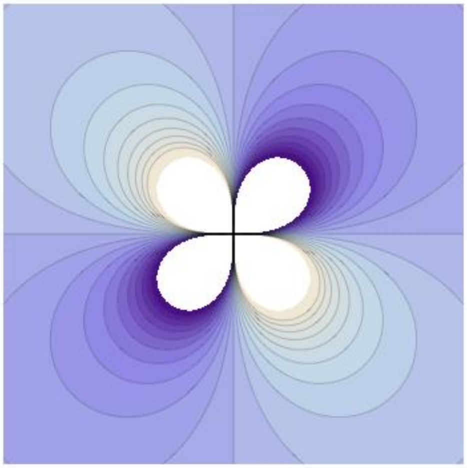

FIGURE 5.2. $1/z^2$

A beautiful rational function which illustrates all of the phenomena we've been talking about is Klein's icosahedral function

$$\frac{(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3}{1728z^5(z^{10}+11z^5-1)^5}$$

which is the degree sixty rational function invariant under the icosahedral symmetries of the Riemann sphere. Witness the eleven poles (plus one at infinity), each of order 5 (two petals per degree). Witness the many zeroes, of order 3 and 2.

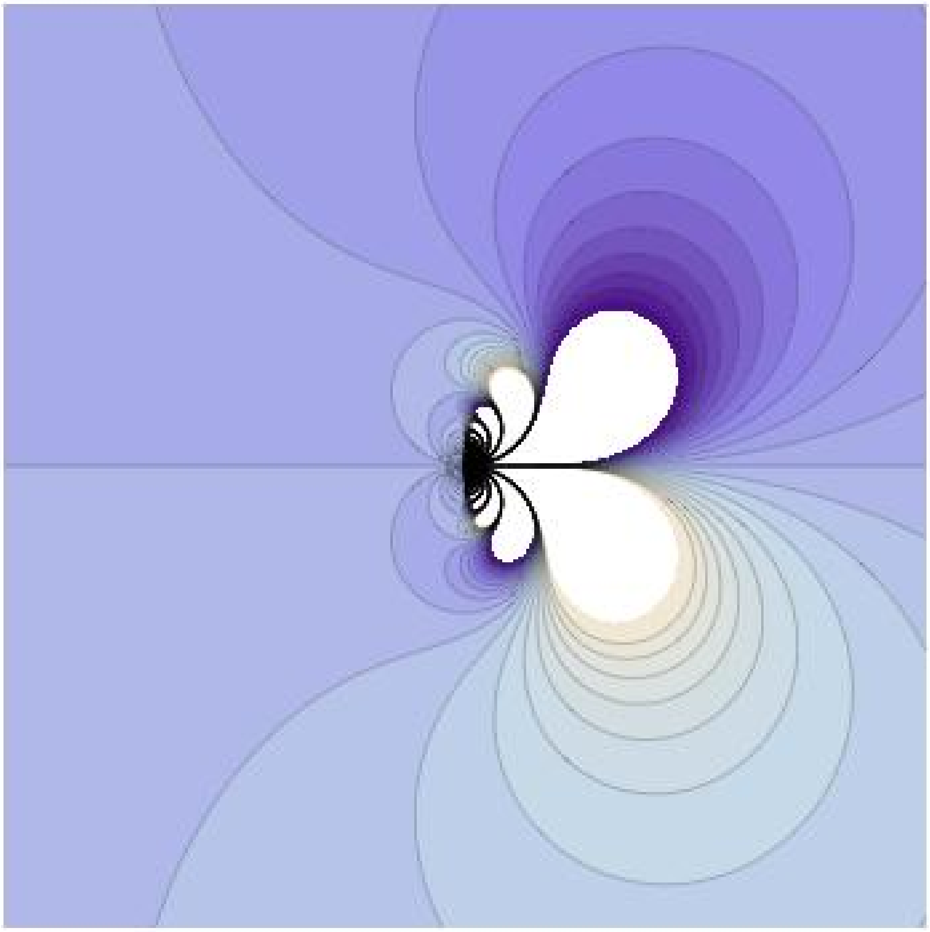

FIGURE 5.3. $e^{1/z}$

6. Oops those were really 1-forms

If we change f by a constant, $f + c$, it has exactly the same contour lines of $u = const$. Thus the picture depends only on the 1-form df. Indeed, given a 1-form α, we may form a local integral $df = \alpha$, determined up to a constant, and plot that.

The Weierstrauss local form construction applies to 1-forms too, and gives that every one form is locally $z^n dz$. We've already seen $d(z^n) = z^{n-1}dz$ for all nonzero integers z in our plots of $f = z^n$. The missing one is $\frac{dz}{z}$, which has no single-valued local antiderivative in the neighborhood of 0. We can nonetheless take its multivalued antiderivative, $\log z = \log r + i\theta$, and plot the lines of constant real and imaginary part.

7. Physical interpretation

The electric field E obeys Maxwell's equations:

$$\nabla \times E = 0 \qquad \nabla E = \rho/\epsilon_0.$$

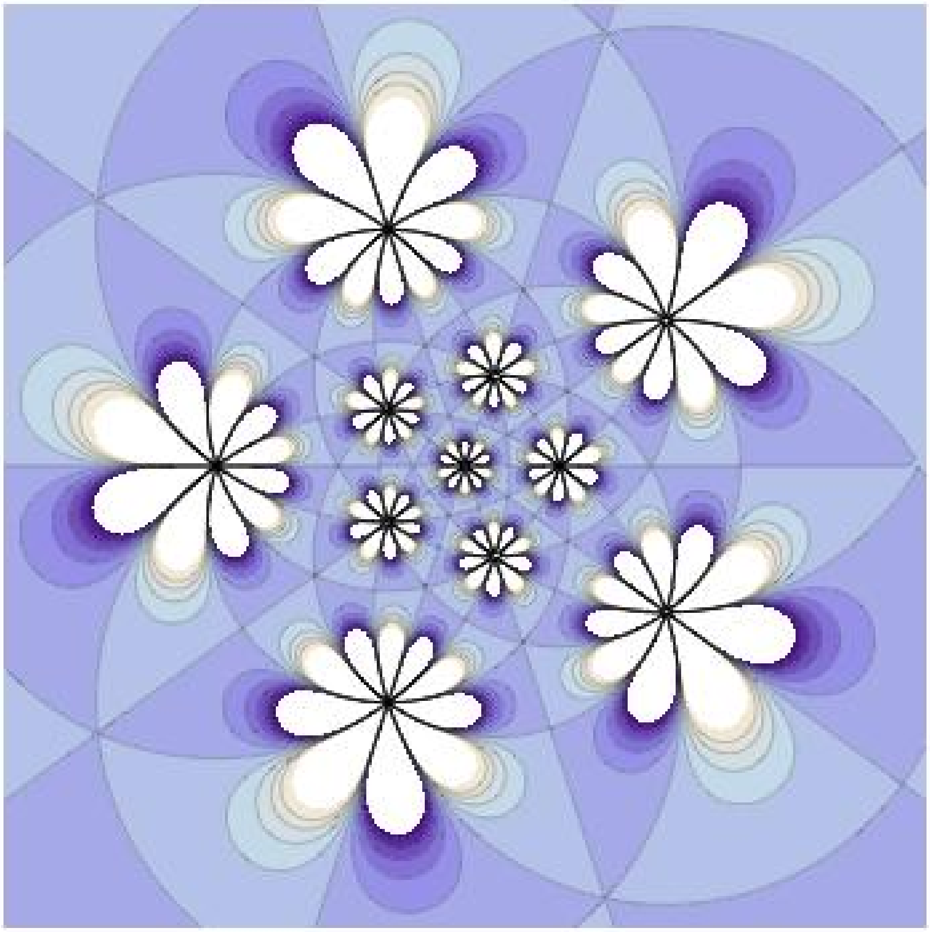

FIGURE 5.4. $$\frac{(-(z^{20}+1)+228(z^{15}-z^5)-494z^{10})^3}{1728z^5(z^{10}+11z^5-1)^5}$$

The electric potential V is harmonic, $\nabla \cdot \nabla V = \nabla \cdot E = 0$, so forms the real part of a holomorphic function. The lines $V = const$ form equipotential lines. The harmonic conjugate of V is the stream function U, whose contours form the *field lines*. For the planar cross-sections to mean anything, the setup needs to have translational symmetry along the z-axis. Then, the electric field is entirely determined by its cross-section, and is determined by a holomorphic function in the plane. For example, a long charged wire generates a radially symmetric field, given by $\log z$. Two charged wires (of strong but opposite charges) brought close together generate a dipole, just as

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(\log z - \log(z-\epsilon)) = \frac{1}{z}.$$

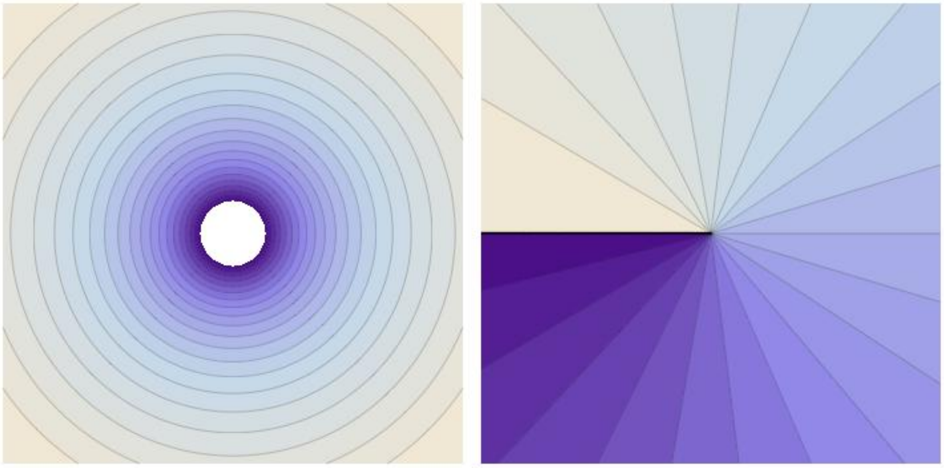

FIGURE 6.1. $\log z$, real and imaginary parts

Another physical model is given by fluid flow. A steady streaming (meaning the course of the fluid doesnt change with time, even though the individual particles move) is given by a vector field X, the velocity at each point in the plane. If the fluid is incompressible (so as much flows into each region as flows out), then $\nabla \cdot X = 0$. If the flow is irrotational (a less reasonable physical assumption), then $\nabla \times X = 0$, and there is some potential V with harmonic conjugate the stream function U. The vanishing of U corresponds to the streamlines of the flow of the fluid.

For example, the function $z + \frac{1}{z}$ has is real on the unit circle (where $z^{-1} = \bar{z}$), so has the unit circle as a streamline. Thus the outside of the unit circle in the picture of the streamlines represents a reasonable approximation for the flow of an incompressible fluid around a cylindrical airfoil. Since harmonic functions are preserved by conformal maps, one can find the flow around any foil by mapping its interior to the unit disk conformally, then pulling back the flow around the unit disk corresponding to $z + \frac{1}{z}$. A popular family of such maps were exploited by Joukowski to give the Joukowski airfoils (named after him)–see http://www.grc.nasa.gov/WWW/K-12/airplane/map.html for a demo by NASA.

8. Do it on a Riemann Surface

We can draw meromorphic 1-forms on arbitary Riemann surfaces in exactly the same we we do on the complex plane. Take local antiderviatves, and draw the lines of constant real (or imaginary) part. Poles and zeroes show up just fine. If you move the 1-form through a family (say the one of the $\mathcal{O}(p+q-r)$ so beloved by Riemann-Roch), you can see extra poles and zeroes cancelling, with Poincaré-Hopf telling us that the total sum of the indices ($-n$ for the singularity at $z^n dz$) must equal $\chi(\Sigma)$ always.

Riemann used the electrical intuition to understand meromorphic functions, at least according to Klein, in *On Riemann's Theory of Algebraic Functions and their Integrals*. (Klein's lecture notes, which is where I learned about this stuff, have really fantastic pictures of deforming 1-forms too.) To get a Riemann map, hook

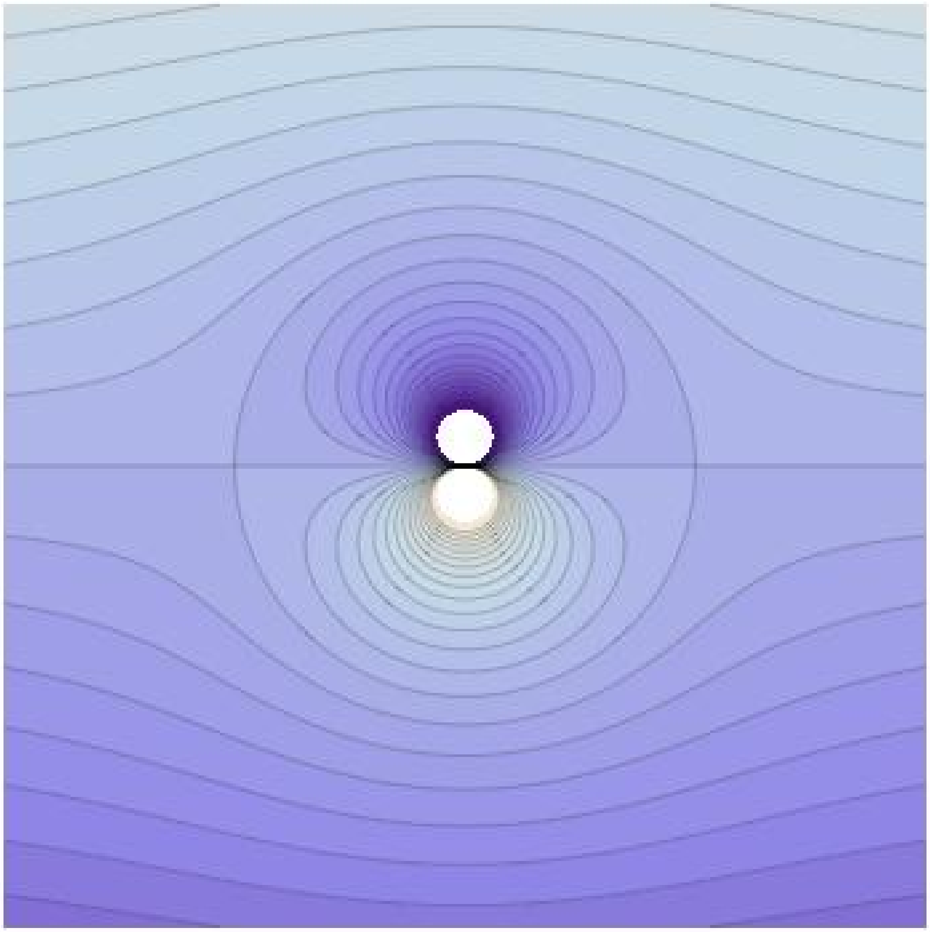

FIGURE 7.1. Flow around a cylinder

up a battery (ie, a dipole) to your metal surface; the harmonic potential it generates is the real part of your desired meromorphic function, which will take you to the Riemann sphere (less a slit). Gzzzzzt and we're done.